IB DP Mathematics: Analysis and Approaches HL
Integral Paper 1 - Markscheme

Question 1 [7 marks]

M1 Attempt at substitution $u = \sin x$ and $du = \cos x\, dx$

M1 Substituting into the integral using $\cos^2 x = 1 - u^2$

A1 $\displaystyle\int \frac{u}{-u^2 + u + 2}\, du$

M1 Partial fraction decomposition: $\displaystyle\frac{u}{(u+1)(2-u)} = \frac{A}{u+1} + \frac{B}{2-u}$

A1 $A = -\frac{1}{3}, \quad B = \frac{2}{3}$

A1 $\displaystyle -\frac{1}{3}\ln|u+1| - \frac{2}{3}\ln|2-u| + C$

A1 **Final answer:** $\displaystyle -\frac{1}{3}\ln(\sin x + 1) - \frac{2}{3}\ln(2 - \sin x) + C$

Question 2 [5 marks]

M1 Combining the integrals: $\displaystyle\int_a^{2a} \left(\frac{e^x}{x} - \frac{e^x}{x^2}\right) dx = 0$

M1 Recognizing the integrand as the derivative of $\frac{e^x}{x}$ (via quotient or product rule)

A1 $\displaystyle\left[\frac{e^x}{x}\right]_a^{2a} = 0 \implies \frac{e^{2a}}{2a} - \frac{e^a}{a} = 0$

A1 $e^{2a} - 2e^a = 0 \implies e^a(e^a - 2) = 0$

A1 As $e^a > 0$, we have $e^a = 2 \implies \boxed{a = \ln 2}$

Question 3 [8 marks]

A1 $dx = a \sec\theta \tan\theta \, d\theta$

A1 Limits: $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{3}$

M1 Substitution into the integral: $\displaystyle\int_{\pi/4}^{\pi/3} \frac{a \sec\theta \tan\theta}{a^3 \sec^3\theta \sqrt{a^2 \sec^2\theta - a^2}} \, d\theta$

A1 Simplification to $\displaystyle\frac{1}{a^3} \int_{\pi/4}^{\pi/3} \cos^2\theta \, d\theta$

M1 Use of double angle identity: $\displaystyle\frac{1}{2a^3} \int_{\pi/4}^{\pi/3} (1 + \cos 2\theta) \, d\theta$

A1 Integration: $\displaystyle\frac{1}{2a^3} \left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3}$

A1 Correct substitution of limits: $\displaystyle\frac{1}{2a^3} \left(\frac{\pi}{3} + \frac{\sqrt{3}}{4} - \frac{\pi}{4} - \frac{1}{2}\right)$

AG **Answer Given:** $\boxed{\dfrac{1}{24a^3}(3\sqrt{3} + \pi - 6)}$

Question 4 [7 marks]

Part (a)

A1 $\displaystyle V = \pi \int_0^{\pi} (e^x - \sin^2 x) \, dx$

Part (b)

M1 $\displaystyle V = \pi \int_0^{\pi} \left[(e^{x/2} + 1)^2 - (\sin x + 1)^2\right] dx$

A1 Expansion: $\displaystyle\pi \int_0^{\pi} (e^x + 2e^{x/2} - \sin^2 x - 2\sin x) \, dx$

M1 Attempt at integration

A1 $\displaystyle\left[e^x + 4e^{x/2} - \left(\frac{x}{2} - \frac{\sin 2x}{4}\right) + 2\cos x\right]_0^{\pi}$

A1 Evaluation: $\displaystyle\pi \left[\left(e^{\pi} + 4e^{\pi/2} - \frac{\pi}{2} - 2\right) - (1 + 4 + 2)\right]$

A1 **Final answer:** $\boxed{\pi\left(e^{\pi} + 4e^{\pi/2} - \dfrac{\pi}{2} - 9\right)}$

Question 5 [7 marks]

Part (a)

A1 $(0,0)$

Part (b)

M1 $\int_0^c \frac{x}{x^2+2}\,dx = \ln 3$

A1 $\left[\frac{1}{2}\ln(x^2+2)\right]_0^c = \ln 3$

M1 Use of logarithm laws

A1 $\frac{1}{2}\ln(c^2+2) - \frac{1}{2}\ln 2 = \ln 3 \implies \ln\left(\frac{c^2+2}{2}\right) = \ln 9$

A1 $\frac{c^2+2}{2} = 9 \implies c^2 = 16$

A1 $\boxed{c = 4}$

Question 6 [10 marks]

Part (a)

M1 First integration by parts with $u = (\ln x)^2$, $dv = dx$

A1 $x(\ln x)^2 - 2\int \ln x\,dx$

M1 Second integration by parts with $u = \ln x$, $dv = dx$

A1 $2(x\ln x - x)$

A1 **Final answer:** $\boxed{x(\ln x)^2 - 2x\ln x + 2x + C}$

Part (b)

M1 Improper integral: $\lim_{k\to 0^+}\left[x(\ln x)^2 - 2x\ln x + 2x\right]_k^1$

A1 At upper limit: $1(0)^2 - 2(0) + 2 = 2$

M1 Consideration of lower limit using L'Hôpital's rule

A1 At lower limit: $\lim_{k\to 0^+}(k(\ln k)^2 - 2k\ln k + 2k) = 0$

A1 **Final answer:** $\boxed{2}$

Question 7 [10 marks]

Part (a)

M1 $u = \cos^{n-1} x,\ dv = \cos x\, dx$
$\implies du = -(n-1)\cos^{n-2} x \sin x\, dx,\ v = \sin x$

A1 $\displaystyle\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\, dx$

M1 Substitute $\sin^2 x = 1 - \cos^2 x$

AG **Answer Given:**
$\displaystyle\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx$

Part (b)

A1 Rearranging: $\displaystyle n\int f_n(x)\, dx = \cos^{n-1} x \sin x + (n-1)\int f_{n-2}(x)\, dx$

AG **Answer Given:**
$\displaystyle\int f_n(x)\, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int f_{n-2}(x)\, dx$

Part (c)

M1 For $n = 4$: $\displaystyle\frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int \cos^2 x\, dx$

A1 For $n = 2$: $\displaystyle\frac{1}{2}\cos x \sin x + \frac{1}{2}x$

A1 $\displaystyle\frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left(\frac{1}{2}\cos x \sin x + \frac{1}{2}x\right) + C$

A1 **Final answer:** $\boxed{\displaystyle\frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + C}$